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A TWO WEIGHT WEAK TYPE INEQUALITY FOR FRACTIONAL INTEGRALS

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ABSTRACT. For $1 < p \leqslant q < \infty$, $0 < \alpha < n$ and $w(x)$, $v(x)$ nonnegative weight functions on R^n we show that the weak type inequality

$$\int_{\{T_\alpha f > \lambda\}} w(x)\,dx \leqslant A\lambda^{-q}\left(\int |f(x)|^p v(x)\,dx\right)^{q/p}$$

holds for all $f \geqslant 0$ if and only if

$$\int_Q [T_\alpha(\chi_Q w)(x)]^{p'} v(x)^{1-p'}\,dx \leqslant B\left(\int_Q w\right)^{p'/q'} < \infty$$

for all cubes Q in R^n. Here T_α denotes the fractional integral of order α, $T_\alpha f(x) = \int |x-y|^{\alpha-n} f(y)\,dy$. More generally we can replace T_α by any suitable convolution operator with radial kernel decreasing in $|x|$.

1. Introduction. Weighted norm inequalities for fractional integrals have been treated by several authors. For example, B. Muckenhoupt and R. L. Wheeden have shown [8] that the one weight strong type inequality

$$(1)\qquad \left(\int_{R^n} |T_\alpha f(x) w(x)|^q\,dx\right)^{1/q} \leqslant C\left(\int_{R^n} |f(x)w(x)|^p\,dx\right)^{1/p} \quad \text{for all } f \geqslant 0$$

where $1/q = 1/p - \alpha/n$ holds if and only if $w(x)^q$ satisfies the A_r condition with $r = 1 + q/p'$. Here $T_\alpha f(x) = \int_{R^n} |x-y|^{\alpha-n} f(y)\,dy$ is the fractional integral or Riesz potential of order α (see [10] for the basic properties of T_α) and the A_r condition on a function $v(x)$ is

$$(A_r)\qquad \left(\int_Q v(x)\,dx\right)^{1/r}\left(\int_Q v(x)^{-r'/r}\,dx\right)^{1/r'} \leqslant C\int_Q dx \quad \text{for all cubes } Q$$

where the second factor on the left side is interpreted as $\|\chi_Q v^{-1}\|_\infty$ in the case $r = 1$. In a different direction, B. Dahlberg [3] has used a capacitary strong type inequality to show that a positive measure ω satisfies the "trace" inequality

$$(2)\qquad \int_{R^n} |T_\alpha f(x)|^p\,d\omega(x) \leqslant C\int_{R^n} |f(x)|^p\,dx \quad \text{for all } f \geqslant 0$$

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if and only if

(3) $$\omega(E) \leqslant A\,\mathrm{Cap}(E) = A \inf\left\{\int |f(x)|^p\,dx;\ T_\alpha f \geqslant 1 \text{ on } E\right\}$$

for all compact subsets E of R^n. See also D. Adams [**1**] and V. Maz'ya [7] for the case α integral. More recently, R. Kerman and the author [**6**] (see also [**9**]) have shown that (2) is equivalent to the simpler condition

(4) $$\int_Q |T_\alpha(\chi_Q\omega)(x)|^{p'}\,dx \leqslant C\int_Q d\omega < \infty \quad \text{for all cubes } Q.$$

However, the characterization of the general two weight strong type inequality for fractional integrals remains open. In this note we address the simpler two weight weak type inequality and give a characterization of it in terms of a condition analogous to (4). As in [**5** and **6**] we will treat operators more general than fractional integrals, namely convolution operators of the form $Tf = K * f$ where $K(x)$ is a positive radial function decreasing in $|x|$. K. Hansson has recently obtained a capacitary strong type inequality for such operators [**5**] and hence the equivalence of (2) and (3) for T in place of T_α (the corresponding equivalence of (2) and (4) is in [**6**]). If μ is a positive measure on R^n we use the notation $|E|_\mu = \int_E d\mu$ and $T(f\mu)(x) = K * (f\mu)(x) = \int K(x-y)f(y)\,d\mu(y)$.

THEOREM. *Suppose* $1 < p \leqslant q < \infty$, ω *and* μ *are positive borel measures on* R^n *and* $Tf = K * f$ *where* $K(x)$ *is a positive lower semicontinuous radial function decreasing in* $|x|$. *If* $n \geqslant 2$ *suppose, in addition, that* $K(x)$ *satisfies* (A_1). *Then the weak type inequality*

(5) $$|\{T(f\mu) > \lambda\}|_\omega \leqslant A\lambda^{-q}\left(\int |f|^p\,d\mu\right)^{q/p} \quad \textit{for all } f \geqslant 0,\ \lambda > 0,$$

holds if and only if

(6) $$\int_Q |T(\chi_Q\omega)|^{p'}\,d\mu \leqslant B|Q|_\omega^{p'/q'} < \infty \quad \textit{for all cubes } Q.$$

Furthermore if A *and* B *are the least such constants, then the ratio* $A^{1/q}/B^{1/p'}$ *is bounded between two positive constants independent of* ω *and* μ.

REMARKS. I. The theorem is also valid for $p = 1$ if (6) is replaced by $\|T(\chi_Q\omega)\|_{L^\infty(\mu)} \leqslant C|Q|_\omega^{1/q'} < \infty$ for all cubes Q.

II. The result stated in the abstract follows from the Theorem with $K(x) = |x|^{\alpha-n}$, $d\omega(x) = w(x)\,dx$, $d\mu(x) = v(x)^{1-p'}\,dx$ and f replaced by $fv^{p'-1}$. Note that $|x|^{\alpha-n}$ satisfies (A_1) for $0 < \alpha < n$.

III. If μ is an A_∞ weight then condition (6) is sufficient for the strong type analogue of (5) (see [**6**]) but, in general, condition (6) is not sufficient. See D. Adams [**1**, Remark 2(iii)] for a counterexample in the case $p = q$ (note that (a'), p. 134 in [**1**] is equivalent to (6) with $d\omega = dx$, $T = T_m$ and p', q' replaced by q, p, respectively) and §3 below for the case $p \leqslant q$.

2. Proof of the Theorem. Assume (5) holds. Provided μ is nontrivial ($0 < |E|_\mu < \infty$ for some set E) the positivity of K together with (5) easily shows that $|Q|_\omega < \infty$ for all cubes Q. The remaining inequality in (6) is an easy consequence of duality (of Lorentz spaces). In fact,

$$\begin{aligned}\left(\int |T(\chi_Q\omega)|^{p'}\,d\mu\right)^{1/p'} &= \sup_{\|f\|_{L^p(\mu)}\leqslant 1} \int T(\chi_Q\omega) f\,d\mu = \sup_{\|f\|_{L^p(\mu)}\leqslant 1} \int_Q T(f\mu)\,d\omega \\ &= \sup_{\|f\|_{L^p(\mu)}\leqslant 1} \int_0^\infty |Q\cap\{T(f\mu)>\lambda\}|_\omega\,d\lambda \\ &\leqslant \int_0^\infty \min\{A\lambda^{-q}, |Q|_\omega\}\,d\lambda \quad \text{by (5)} \\ &= q'A^{1/q}|Q|_\omega^{1/q'}\end{aligned}$$

and so (6) holds with $B \leqslant (q')^{p'}A^{p'/q}$.

Conversely, suppose (6) holds and, without loss of generality, that f is nonnegative with compact support and satisfies $\int |f|^p\,d\mu < \infty$. The main idea of the proof is to establish a "good λ inequality" (in much the same manner as is done in R. Coifman [**2**]) for Tf relative to the maximal operator

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|_\omega}\int_Q T(\chi_Q f)\,d\omega.$$

We begin with the case $n = 1$. Fix $0 < \beta < 1$ and $\lambda > 0$. Since $Tf\mu$ is lower semicontinuous we can write $\{Tf\mu > \lambda\} = \dot{\bigcup}_k I_k$ where the intervals $I_k = (a_k, b_k)$ are disjoint. Moreover, (6) implies that $|I_k|_\omega < \infty$ for all k (if I_k is infinite, then $\lim_{x\to\infty} K(x) > 0$ and it is easy to see that (6) implies $\int_{-\infty}^{\infty} d\omega < \infty$). We now discard those I_k with $|I_k|_\omega = 0$ and denote by F the set of indices k such that

$$\frac{1}{|I_k|_\omega}\int_{I_k} T(\chi_{I_k} f\mu)\,d\omega > \beta\lambda \tag{7}$$

and by G the set of k for which (7) fails. For k in F we have

$$\begin{aligned}\lambda^q |I_k|_\omega &< \beta^{-q}|I_k|_\omega^{1-q}\left(\int_{I_k} T(\chi_{I_k} f\mu)\,d\omega\right)^q \\ &= \beta^{-q}|I_k|_\omega^{1-q}\left(\int_{I_k} T(\chi_{I_k}\omega) f\,d\mu\right)^q \\ &\leqslant \beta^{-q}|I_k|_\omega^{1-q}\left(\int_{I_k} T(\chi_{I_k}\omega)^{p'}\,d\mu\right)^{q/p'}\left(\int_{I_k} |f|^p\,d\mu\right)^{q/p} \\ &\leqslant \beta^{-q}B^{q/p'}\left(\int_{I_k} |f|^p\,d\mu\right)^{q/p} \quad \text{by (6).}\end{aligned} \tag{8}$$

Now observe that if $\tilde{I}_k$ denotes the complement of I_k then $T(\chi_{\tilde{I}_k} f\mu) \leqslant 2\lambda$ on I_k by the maximum principle. Indeed, if x is in I_k then

$$\begin{aligned} T(\chi_{\tilde{I}_k} f\mu)(x) &= \left(\int_{-\infty}^{a_k} + \int_{b_k}^{\infty}\right) K(x-y) f(y)\, d\mu(y) \\ &\leqslant \int_{-\infty}^{a_k} K(a_k - y) f(y)\, d\mu(y) + \int_{b_k}^{\infty} K(b_k - y) f(y)\, d\mu(y) \\ &\leqslant Tf(a_k) + Tf(b_k) \leqslant 2\lambda. \end{aligned}$$

Thus for k in G

$$\text{(9)} \qquad \begin{aligned} |I_k \cap \{T(f\mu) > 3\lambda\}|_\omega &\leqslant \left|I_k \cap \left\{T(\chi_{I_k} f\mu) > \lambda\right\}\right|_\omega \\ &\leqslant \frac{1}{\lambda} \int_{I_k} T(\chi_{I_k} f\mu)\, d\omega \leqslant \beta |I_k|_\omega \end{aligned}$$

since (7) fails. Combining (8) and (9) we obtain the "good λ inequality"

(10)

$$\begin{aligned} (3\lambda)^q |\{T(f\mu) > 3\lambda\}|_\omega &= \sum_k (3\lambda)^q |I_k \cap \{T(f\mu) > 3\lambda\}|_\omega \\ &\leqslant 3^q \sum_{k \in F} \lambda^q |I_k|_\omega + 3^q \lambda^q \sum_{k \in G} |I_k \cap \{T(f\mu) > 3\lambda\}|_\omega \\ &\leqslant \left(\frac{3}{\beta}\right)^q B^{q/p'} \sum_{k \in F} \left(\int_{I_k} |f|^p\, d\mu\right)^{q/p} + 3^q \beta \lambda^q \sum_{k \in G} |I_k|_\omega \\ &\leqslant \left(\frac{3}{\beta}\right)^q B^{q/p'} \left(\int |f|^p\, d\mu\right)^{q/p} + 3^q \beta \lambda^q |\{T(f\mu) > \lambda\}|_\omega \end{aligned}$$

since $q/p \geqslant 1$. Choose $\beta = \frac{1}{2}(\frac{1}{3})^q$ and take the supremum in (10) over $0 < \lambda \leqslant t/3$ to obtain

(11)

$$\sup_{0<\lambda\leqslant t} \lambda^q |\{T(f\mu) > \lambda\}|_\omega \leqslant \frac{3^{q^2+q}}{2^q} B^{q/p'} \left(\int |f|^p\, d\mu\right)^{q/p} + \frac{1}{2} \sup_{0<\lambda\leqslant t} \lambda^q |\{T(f\mu) > \lambda\}|_\omega$$

for all $t > 0$. If we can show that the left side of (11) is finite for all $t > 0$, we can subtract the second term on the right side of (11) from both sides to obtain (5) with $A \leqslant 3^{q^2+q} B^{q/p'}/2^{q-1}$. To see that the left side of (11) is finite suppose that f is supported in an interval $I = (a, b)$ and that $r > 2b - a$. Then from (6) we have

$$B\left(\int_a^r d\omega\right)^{p'/q'} \geqslant \int_a^b |T(\chi_{(a,r)}\omega)(x)|^{p'}\, d\mu(x) \geqslant \int_I \left|K(r-x) \int_a^r d\omega\right|^{p'} d\mu(x)$$

and so $(\int_I K(r-x)^{p'}\,d\mu(x))(\int_a^r d\omega)^{p'/q} \leq B$. Thus for $r > 2b - a$ and $\lambda = T(f\mu)(r)$ we have

$$\lambda^q|\{T(f\mu) > \lambda\} \cap (a,\infty)|_\omega \leq \left(\int_I K(r-x)f(x)\,d\mu(x)\right)^q \int_a^r d\omega$$

$$\leq \left(\int_I f^p\,d\mu\right)^{q/p}\left(\int_I K(r-x)^{p'}\,d\mu(x)\right)^{q/p'}\int_a^r d\omega$$

$$\leq \left(\int_I f^p\,d\mu\right)^{q/p} B^{q/p'} < \infty.$$

Similarly one can show that for λ sufficiently small

$$\lambda^q|\{T(f\mu) > \lambda\} \cap (-\infty, b)|_\omega \leq \left(\int_I f^p\,d\mu\right)^{q/p} B^{q/p'} < \infty$$

and this shows that the left side of (11) is finite for $t > 0$ and completes the proof of the Theorem in the case $n = 1$.

We now turn to the case $n \geq 2$ and assume that K satisfies the (A_1) condition. Recall that f is nonnegative with compact support and satisfies $\int |f|^p\,d\mu < \infty$. Again fix $0 < \beta < 1$ and $\lambda > 0$. Let Mf denote the maximal function of f, i.e.

$$Mf(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f|$$

and for $r > 0$ denote by rQ the cube concentric with Q and having r times the side length of Q. Using a variant of the Whitney covering lemma in C. Fefferman [**4**, p. 16] we can write $\Omega_\lambda = \{M(T(f\mu)) > \lambda\} = \dot{\bigcup}_k Q_k$ where the cubes Q_k satisfy both a Whitney condition

$$(12) \qquad rQ_k \cap \tilde{\Omega}_\lambda \neq 0 \quad \text{for all } k$$

and a finite overlap condition

$$(13) \qquad \sum_k \chi_{2Q_k} \leq D\chi_{\Omega_\lambda}$$

where r and D are positive constants depending only on the dimension n.

Since K satisfies (A_1) and is decreasing as a function of $|x|$ it is easy to show that

$$(14) \qquad K(x) \leq CK(y) \quad \text{for } |y| \leq 2|x|.$$

Fix k for the moment and let $f_1 = f\chi_{2Q_k}$ and $f_2 = f - f_1$. For x in Q_k we have

$$T(f_2\mu)(x) = \int_{y\notin 2Q_k} K(x-y)f(y)\,d\mu(y)$$

$$\leq C'\int_{y\notin 2Q_k}\left(\frac{1}{|Q_k|}\int_{Q_k} K(z-y)\,dz\right)f(y)\,d\mu(y) \quad \text{by (14)}$$

$$\leq \frac{C'}{|Q_k|}\int_{Q_k} T(f\mu)(z)\,dz$$

$$\leq C'r^n\frac{1}{|rQ_k|}\int_{rQ_k} T(f\mu)(z)\,dz \leq C'r^n\lambda \quad \text{by (12).}$$

Thus for $\gamma > 2C'r^n$ we have

$$\{T(f\mu) > \gamma\lambda\} \cap Q_k \subset \{T(\chi_{2Q_k} f\mu) > \tfrac{1}{2}\gamma\lambda\} \cap Q_k. \tag{15}$$

Denote by F the set of indices k such that

$$\frac{1}{|2Q_k|_\omega} \int_{2Q_k} T(\chi_{2Q_k} f\mu)\, d\omega > \beta\lambda \tag{16}$$

and by G the set of k for which (16) fails. Using (15) and arguing as in the case $n = 1$ (see (8) and (9)) we obtain

$$|\{T(f\mu) > \gamma\lambda\} \cap Q_k|_\omega \leqslant \max\left\{\beta|2Q_k|_\omega, (\beta\lambda)^{-q} B^{q/p'}\left(\int_{2Q_k} f^p\, d\mu\right)^{q/p}\right\}$$

for all k and then summing over k and using (13) we get

$$|\{T(f\mu) > \gamma\lambda\}|_\omega \leqslant \beta D|\Omega_\lambda|_\omega + (\beta\lambda)^{-q} B^{q/p'}\left(D\int f^p\, d\mu\right)^{q/p} \tag{17}$$

for $\gamma > 2C'r^n$ and $0 < \beta < 1$. However $T(f\mu)$ satisfies (A_1) with the same constant C that works for K and so $M(T(f\mu)) \leqslant CT(f\mu)$. Thus $|\Omega_\lambda|_\omega \leqslant |\{T(f\mu) > \lambda/C\}|_\omega$ and if we use this inequality on the first term on the right side of (17) we obtain an analogue of the good λ inequality (10) and the proof can now be completed as in the case $n = 1$.

3. An example. Fix $p = 2 \leqslant q < \infty$, $n = 1$ and $1/2 \leqslant \alpha + 1/q < 1$ with $\alpha > 0$. We construct a pair of weights w, v on the real line satisfying the condition in the abstract ($p = 2$)

$$\int_Q [T_\alpha(\chi_Q w)]^2 v^{-1} \leqslant B\left(\int_Q w\right)^{2/q'} \quad \text{for all intervals } Q \tag{18}$$

but not the corresponding strong type inequality

$$\left(\int |T_\alpha f|^q w\right)^{1/q} \leqslant C\left(\int |f|^2 v\right)^{1/2} \quad \text{for all } f \geqslant 0. \tag{19}$$

Let

$$f(x) = x^{-1}|\log x|^{-q'}\chi_{(0,1/2)}(x)$$

and set $v(x) = f(x)^{-1}$ and $w(x) = x^{q-q\alpha-1}|\log x|^{q'-1}\chi_{(0,1/2)}(x)$. Since $T_\alpha f(x) \approx x^{\alpha-1}|\log x|^{1-q'}$ we have that the left side of (19) is infinite while the right side is finite. On the other hand, using the estimates

$$\|T_\alpha(\chi_{(a,r)} w)\|_\infty \lesssim (r-a)^\alpha r^{q-q\alpha-1}|\log r|^{q'-1}$$

and

$$\int_a^r x^{-1}|\log x|^{-q'}\, dx \lesssim \begin{cases} |\log r|^{1-q'}, & 0 \leqslant a < r/2, \\ |\log r|^{-q'}\left(\dfrac{r-a}{r}\right), & r/2 \leqslant a < r, \end{cases}$$

which are valid for $1/2 \leqslant \alpha + 1/q < 1$, $\alpha > 0$ and $0 \leqslant a < r \leqslant 1/2$ we obtain

$$\int_a^r \left[T_\alpha(\chi_{(a,r)}w)\right]^2 v^{-1} \lesssim (r-a)^{2/q'} r^{2q+2\alpha-2q\alpha-2-2/q'} |\log r|^{2-2/q'} \lesssim \left(\int_a^r w\right)^{2/q'}$$

for $0 \leqslant a < r \leqslant 1/2$ which is (18).

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